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Procedia IUTAM 8 (2013) 13 – 21
Procedia IUTAM
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IUTAM Symposium on Waves in Fluids: Effects of Nonlinearity, Rotation, Stratification and Dissipation

Stability of Swirl Axisymmetric Incompressible Flow

Sergey P. Aktershev *, Pavel A. Kuibin

Kutateladze Institute of Thermophysics SB RAS, Lavrentyev av., 1, Novosibirsk, 630090, Russia

Abstract

A new method of solution of the stability problem for swirl flow of a viscous incompressible liquid is developed. The method is based on the expansion of required functions into power series of radial coordinate and allows to avoid numerical integration of the system of differential equations with a singular point. As an example the stability of Poiseuille flow in a rotating pipe is considered. Calculations by new method agree well with known numerical integration result.

Selection and/or peer-review under responsibility of Yuli Chashechkin and David Dritschel

Keywords: incompressible liquid, axisymmetric swirl flow, stability, solution technique

1. Introduction

The swirl flows are widely applied in practice. In high-head water turbines the flow swirling is used for twisting moment creation on the runner and in an optimum regime on an exit from the runner the swirling is practically absent. In regimes of partial or forced loading considerable level of swirling remains in a wake behind runner. Arising problems are general for various vortex apparatuses for heat transfer intensification, for chemical reactions, in bioreactors and separators, in vortex torches, etc. For the purpose of intensification the vortex apparatuses operates in conditions of high swirling. As a result, instability often develops; the flow becomes three-dimensional and unsteady. In particular, there can be regular pulsations of pressure at frequencies close to eigen mechanical or acoustical frequencies of devices and apparatuses. At present time the methods of computational fluid dynamics are widely applied to flows in technological devices. Nevertheless, for the adequate description of processes the a priori information about possible instabilities is extremely actual. At the same time the analysis of stability allows to predict without conducting of expensive and long calculations the ranges of most unstable frequencies (lengths of waves), and also regime characteristics in which the flow is unstable. At the linear stability analysis of swirl flows the Navier – Stokes equations are linearized and exponential growing (or damping) of perturbations is considered in the space or temporal statement. The problem is reduced to search of eigen frequen-

* Corresponding author. Tel.: +7-383-330-60-44; fax: +7-383-330-84-80.
E-mail address: sergey-aktershev@mail.ru

Selection and/or peer-review under responsibility of Yuli Chashechkin and David Dritschel
doi:10.1016/j.piutam.2013.04.003

cies (lengths of waves) and appropriate eigenfunctions for various oscillation modes. In addition to the standard difficulties in problem of hydrodynamic stability due to presence of small coefficient at a highest derivative (for large Reynolds numbers) and bifurcations, for swirl flow there is one more difficulty − singularity on flow axis (and for unbounded flow additional singularity on infinity). Near to singular points the eigenfunctions for different modes are close to each other and at numerical integration the "jump" from one mode to another is possible. To avioid these difficulties near to singular points the asymptotic solutions by Frobenius method have being constructed (see, for example, [1]) that allows to transfer the boundary conditions into some regular points, and orthogonality procedure is applied for separation of the linearly independent solutions.

In this paper a new solution method of the stability problem of the swirl flow is developed. The method is based on the expanding of required functions into series on powers of radial coordinate. Presented method allows avoiding the difficulties inherent to numerical integration of system of the differential equations with a singular point.

Nomenclature

r, φ, z	cylindrical coordinates	$\mathrm{Re}=U_0R/\nu$	Reynolds number
$c = c_r+ic_i$	complex velocity	U_0	velocity on an axis
n	azimuthal number (integer)	$F(r)$	amplitude of axial component
α	wavenumber (real)	$H(r)$	amplitude of azimuthal component
V_r, V_φ, V_z	velocity components	$iS(r)$	amplitude of radial component
R	radius of pipe	$P(r)$	amplitude of pressure

2. The linear analysis of stability of bounded flow

Consider axisymmetric swirl flow of incompressible liquid in a pipe with the velocity components written in cylindrical coordinates as $V_z = U(r)$, $V_r = 0$, $V_\varphi = W(r)$. We take the radius of pipe R as the length scale, velocity on an axis U_0 velocity scale, R/U_0 as the time scale and turn to dimensionless variables, keeping traditional designations for all variables. We take perturbations of velocity components and pressure in the form of a traveling wave

$$\left\{\overline{V}_z,\overline{V}_r,\overline{V}_\phi,\overline{p}\right\}=\left\{F,iS,H,P\right\}\exp\left(i(\alpha z+n\phi-\alpha ct)\right) \tag{1}$$

Expression (1) describes the growing or damping perturbations, periodic on z and φ . Here i is the imaginary unit, α is the real wavenumber, n is the azimuthal wavenumber ($n = 0, \pm 1, \pm 2, \ldots$), $c = c_r + ic_i$ is the complex velocity of propagating perturbation, $F(r)$, $S(r)$, $H(r)$, $P(r)$ are the complex amplitude functions. Positive values of n correspond to wave propagating in a swirling direction, negative – for opposite direction. The magnitude of αc_i gives temporary increment, αc_r is the frequency of oscillations. Positive values of the increment correspond to growing perturbations.

Substitution of (1) into the linearized Navier – Stokes equations gives the system of the differential equations [1]

$$\frac{1}{i\,\mathrm{Re}}\Big(r(rS')'-(n^2+1)S-2nH\Big)=r^2\left(\gamma+\alpha^2/i\,\mathrm{Re}\right)S+2rHW-r^2P' \tag{2}$$

$$\frac{1}{i\,\mathrm{Re}}\Big(r(rH')'-(n^2+1)H-2nS\Big)=\left(\gamma+\alpha^2/i\,\mathrm{Re}\right)r^2H+(W'+W/r)r^2S+rnP \tag{3}$$

$$\frac{1}{i\,\mathrm{Re}}\left(r(rF')' - n^2F\right) = \left(\gamma + \alpha^2 / i\,\mathrm{Re}\right)r^2F + \alpha r^2P + r^2SU' \tag{4}$$

$$\alpha rF + (rS)' + nH = 0 \tag{5}$$

Here the prime means a derivative with respect to r, $\mathrm{Re} = U_0R/\nu$ is the Reynolds number, parameter $\gamma = \alpha(U - c) + nW / r$. Boundary conditions are as follows. At the pipe axis: at $n = 0$ $F(0)$ and $P(0)$ are restricted, $S(0) = H(0) = 0$; at $n = \pm 1$ $F(0) = P(0) = 0$, $S(0) \pm H(0) = 0$; at $|n| > 1$ $F(0) = P(0) = S(0) = H(0) = 0$. On a wall: $S(1) = H(1) = F(0) = 0$. The solution of the equations (2) – (5) with the mentioned boundary conditions exists only for certain values of complex velocity (eigenvalue problem).

Let's introduce variables $\psi = (S + H) / 2$, $\theta = (S - H) / 2$ and rewrite the system (2) – (5) with new variables:

$$\frac{1}{i\,\mathrm{Re}}\left(r(r\psi')' - (n+1)^2\psi\right) = \left(\gamma + \frac{\alpha^2}{i\,\mathrm{Re}} + \beta_1\right)r^2\psi + \beta_2 r^2\theta + \frac{1}{2}\left(rnP - r^2P'\right) \tag{6}$$

$$\frac{1}{i\,\mathrm{Re}}\left(r(r\theta')' - (n-1)^2\theta\right) = \left(\gamma + \frac{\alpha^2}{i\,\mathrm{Re}} - \beta_1\right)r^2\theta - \beta_2 r^2\psi - \frac{1}{2}\left(rnP + r^2P'\right) \tag{7}$$

$$\frac{1}{i\,\mathrm{Re}}\left(r(rF')' - n^2F\right) = \left(\gamma + \frac{\alpha^2}{i\,\mathrm{Re}}\right)r^2F + \alpha r^2P + r^2U'\left(\psi + \theta\right) \tag{8}$$

$$r\theta' + r\psi' + (n+1)\psi - (n-1)\theta = -\alpha rF \tag{9}$$

Here we introduced two auxiliary functions: $\beta_1(r) = 0{,}5\left(W' + 3W / r\right)$ and $\beta_2(r) = 0{,}5\left(W' - W / r\right)$. Boundary conditions for ψ and θ are as follows. On a pipe axis: at $n = 0$ $\psi(0) = \theta(0) = 0$; at $n = 1$ $\psi(0) = 0$, $\theta(0) = S(0)$; at $n = -1$ $\psi(0) = S(0)$, $\theta(0) = 0$; at $|n| > 1$ $\psi(0) = \theta(0) = 0$. On a wall: $\psi(1) = \theta(1) = F(1) = 0$.

Let's notice, that unlike (2) – (5), the system (6) – (9) possesses property of symmetry, in particular, equations are invariant relative the replacement $n \to -n$, $\psi \to \theta$, $\theta \to \psi$, $\beta_1 \to -\beta_1$, $\beta_2 \to -\beta_2$.

3. Stability of Poiseuille flow in a rotating pipe

As an example we consider a flow in a rotating pipe with Poiseuille distribution of axial velocity $U = 1 - r^2$ and the azimuthal velocity distributed by the law of solid-state rotation $W = qr$. Here $q = \Omega R/U_0$ is the swirl parameter, Ω is the angular speed of pipe rotation. Considered flow in a rotating pipe unlike the classical Poiseuille flow is unstable to non-axisymmetric perturbations. For the first time this fact has been discovered in [2, 3] by asymptotic methods, later it has been confirmed in [4 – 6] by numerical methods. Depending on parameters of a problem (Reynolds number and swirl parameter) the various growing modes can exist simultaneously. As well as for free vortex, the mode $n = -1$ is most unstable. In [5] the curves of a neutral stability were calculated, critical values of Reynolds numbers and swirling parameter were defined, comparison with experimental data was done. Detailed research of this problem is carried out in [1] by means of numerical integration of the equations (2)–(5). Calculations in [1] consisted of several stages. Near to a singular point $r = 0$ the asymptotic solutions by Frobenius method was constructed that allowed setting boundary conditions at $r = r_a > 0$. On the interval $r_a \le r \le 1$

the solution proceeded by numerical integration. Selection of eigenvalues c_r, c_i by means of the iterative procedure provided fulfilling of the boundary conditions $S(1) = H(1) = F(0) = 0$. The system (2) – (5) was represented in the form of six differential equations of the first order and was integrated by the Kutta – Merson method. Main difficulty in realization of this method consists in fast (parasitic) growth of one or several solutions at numerical integration therefore their linear independence disappear. To avoid these difficulties the orthogonalization procedure for linearly independent solutions was applied on each integration step.

Let's consider a method of solution of equations (6) – (9) without numerical integration and allowing to avoid the difficulties inherent to it. For Poiseuille flow in rotating pipe $\gamma = \alpha(1-c) + nq - \alpha r^2$, $\beta_1(r) = 2q$, $\beta_2(r) = 0$, therefore the first brackets in a right member of the equations (6) and (7) can be written as

$$\left(\gamma + \alpha^2 / i\,\mathrm{Re} + \beta_1\right) = \alpha(1-c) + nq + \alpha^2 / i\,\mathrm{Re} - \alpha r^2 + 2q$$

$$\left(\gamma + \alpha^2 / i\,\mathrm{Re} - \beta_1\right) = \alpha(1-c) + nq + \alpha^2 / i\,\mathrm{Re} - \alpha r^2 - 2q$$

Rescaling $\psi = i\,\mathrm{Re}\,\tilde{\psi}$, $\theta = i\,\mathrm{Re}\,\tilde{\theta}$, $F = i\,\mathrm{Re}^2\,\tilde{F}$, $\alpha = \tilde{\alpha} / \mathrm{Re}$, $q = \tilde{q} / \mathrm{Re}$ is obliged by necessity for considering high values of Reynolds number. It is known, that at $\mathrm{Re} \to \infty$ the wavenumber of growing perturbations has an order 1/Re, therefore $\tilde{\alpha} \sim 1$. For variables with a tilde equations (6) – (9) obtain the following view

$$r(r\tilde{\psi}')' - (n+1)^2\tilde{\psi} = \left(b + 2i\tilde{q} - i\tilde{\alpha}r^2\right)r^2\tilde{\psi} + \left(rnP - r^2P'\right)/2 \tag{10}$$

$$r(r\tilde{\theta}')' - (n-1)^2\tilde{\theta} = \left(b - 2i\tilde{q} - i\tilde{\alpha}r^2\right)r^2\tilde{\theta} - \left(rnP + r^2P'\right)/2 \tag{11}$$

$$r(r\tilde{F}')' - n^2\tilde{F} = \left(b - i\tilde{\alpha}r^2\right)r^2\tilde{F} + r^2P\tilde{\alpha} / \mathrm{Re}^2 - 2ir^3(\tilde{\psi} + \tilde{\theta}) \tag{12}$$

$$r\tilde{\theta}' + r\tilde{\psi}' + (n+1)\tilde{\psi} - (n-1)\tilde{\theta} = -\tilde{\alpha}r\tilde{F} \tag{13}$$

Here $b = i\left(\tilde{\alpha} + n\tilde{q} - \tilde{\alpha}c_r\right) + \tilde{\alpha}c_i + \tilde{\alpha}^2 / \mathrm{Re}^2$.

Now expand functions $\tilde{\psi}(r)$, $\tilde{\theta}(r)$, $\tilde{F}(r)$, $P(r)$ into power series of radial coordinate r, i.e. present them as

$$\tilde{\psi} = \psi_0 r^k + \psi_1 r^{k+2} + \psi_2 r^{k+4} + \ldots + \psi_m r^{k+2m},\ \tilde{\theta} = \theta_0 r^k + \theta_1 r^{k+2} + \theta_2 r^{k+4} + \ldots + \theta_m r^{k+2m} \tag{14}$$

$$\tilde{F} = f_1 r^{k+1} + f_2 r^{k+3} + \ldots + f_m r^{k+2m-1},\ P = p_1 r^{k+1} + p_2 r^{k+3} + \ldots + p_m r^{k+2m-1} \tag{15}$$

For $\tilde{\psi}$ and $\tilde{\theta}$ series begin with $k = |n| - 1$, and for $\tilde{F}$ and P with $|n|$. At $k = 0$ the coefficients ψ_0 and θ_0 equal to values $\psi(0)$ and $\theta(0)$, and for $|n| > 1$ the boundary conditions $F(0) = P(0) = \psi(0) = \theta(0) = 0$ are satisfied. Further we will consider only negative modes ($n < 0$) as only they give instability.

Substituting series (14), (15) in the equations (10) – (13) and equating coefficients at identical degrees of r in both parts, we receive recurrent relations for them:

$$\left((k+2m)^2 - k^2\right)\psi_m = \left(b + 2i\tilde{q}\right)\psi_{m-1} - i\tilde{\alpha}\psi_{m-2} - p_m\left(k+m\right) \tag{16}$$

$$\left((k+2m)^2 - (k+2)^2\right)\theta_m = \left(b - 2i\tilde{q}\right)\theta_{m-1} - i\tilde{\alpha}\theta_{m-2} - p_m\left(m-1\right) \tag{17}$$

$$\left((k+2m-1)^2 - (k+1)^2\right)f_m = bf_{m-1} - i\tilde{\alpha}f_{m-2} + p_{m-1}\tilde{\alpha} / \mathrm{Re}^2 - 2i\left(\psi_{m-2} + \theta_{m-2}\right) \tag{18}$$

$$2\left(m+k+1\right)\theta_m + 2m\psi_m = -\tilde{\alpha}f_m. \tag{19}$$

These relations written for $m = 0, 1, 2$ …, allow to express coefficients of series with number m through the previous ones with numbers $m - 1$ and $m - 2$.

If "starting" coefficients with numbers 0 and 1 are known the remain ones can be calculated from recurrent relations, thereby, all required functions will be received. Relations (16) – (19) can be easy solved for ψ_m, θ_m, p_m, f_m. After some transformations we have

$$f_m = \frac{bf_{m-1} - i\tilde{\alpha} f_{m-2} + p_{m-1}\tilde{\alpha} / \mathrm{Re}^2 - 2i\left(\psi_{m-2} + \theta_{m-2}\right)}{4(m+k)\left(m-1\right)} \tag{20}$$

$$p_m = \frac{4i\tilde{q}E_{m-1} - 2i\tilde{\alpha}\left(m\psi_{m-2} + (m+k+1)\theta_{m-2} + \tilde{\alpha} f_{m-2} / 2\right) + p_{m-1}\tilde{\alpha}^2 / \mathrm{Re}^2}{4(m+k)(m-1)} \tag{21}$$

$$\theta_m = -\frac{E_m + \tilde{\alpha} f_m / 2}{2(m+k+1)}, \ \psi_m = \frac{E_m - \tilde{\alpha} f_m / 2}{2m} \tag{22}$$

$$E_m = \frac{bE_{m-1} - i\tilde{\alpha}\left(\tilde{q} f_{m-1} + (m-1)\psi_{m-2} - (m+k)\theta_{m-2}\right)}{4(k+m)(m-1)} \tag{23}$$

Here $E_m = m\psi_m - (m+k+1)\theta_m$.

From (20) – (22) it follows $\theta_0 = 0$, and coefficients p_1, f_1, ψ_0 remains not defined. Thus, p_1, f_1, ψ_0 are the "starting" coefficients from which all remaining ones can be calculated. In particular, for coefficients with number 1 we have from (16) and (19):

$$\psi_1 = \frac{1}{4}\left(\frac{b+2i\tilde{q}}{k+1}\psi_0 - p_1\right), \ \theta_1 = -\frac{\psi_1 + \tilde{\alpha} f_1 / 2}{k+2} = \frac{1}{4(k+2)}\left(p_1 - \frac{b+2i\tilde{q}}{k+1}\psi_0 - 2\tilde{\alpha} f_1\right)$$

From here we receive $E_1 = \psi_1 - (k+2)\theta_1 = \frac{1}{2}\left(\frac{b+2i\tilde{q}}{k+1}\psi_0 + \tilde{\alpha} f_1 - p_1\right)$.

"Starting" coefficients are calculated using boundary conditions on a wall. Let us take enough terms of series N. Taking series (14), (15) at $r = 1$ we have

$$\tilde{\theta}(1) = \theta_0 + \theta_1 + \theta_2 + \ldots + \theta_N, \ \tilde{F}(1) = f_1 + f_2 + \ldots + f_N.$$

From here it follows, that $\tilde{F}(1)$ and $\tilde{\theta}(1)$ are determined through the "starting" coefficients. As recurrent relationships are linear, $\tilde{F}(1)$ and $\tilde{\theta}(1)$ are linear functions of "starting" coefficients, i.e. they can be presented as

$$\tilde{F}(1) = A_1 p_1 + B_1 f_1 + D_1\psi_0, \ \tilde{\theta}(1) = A_2 p_1 + B_2 f_1 + D_2\psi_0$$

Unknown constants A_1, A_2, B_1, B_2, D_1, D_2 we find with the following algorithm. If we put $\left(p_1, f_1, \psi_0\right) = \left(0,0,1\right)$, then by recurrence formulae we can calculate other coefficients of series (14), (15). Thereby we gain values $\tilde{F}(1)$ and $\tilde{\theta}(1)$ which are equal accordingly to D_1 and D_2. Setting $\left(p_1, f_1, \psi_0\right) = \left(1,0,0\right)$ yields that $\tilde{F}(1)$ and $\tilde{\theta}(1)$ are equal A_1 and A_2. Finally setting $\left(p_1, f_1, \psi_0\right) = \left(0,1,0\right)$ we find $\tilde{F}(1)$ and $\tilde{\theta}(1)$ that are equal B_1 and B_2. Boundary conditions $\tilde{F}(1) = \tilde{\theta}(1) = 0$ give system of linear equations

$$A_1 p_1 + B_1 f_1 + D_1\psi_0 = 0, \ A_2 p_1 + B_2 f_1 + D_2\psi_0 = 0 \tag{24}$$

As the functions $\tilde{\psi}(r)$, $\tilde{\theta}(r)$, $\tilde{F}(r)$, $P(r)$ are determined within an arbitrary factor, the value ψ_0 can be an arbitrary constant. Thus, from (24) we have

$$p_1 = \psi_0\left(D_2B_1 - D_1B_2\right)/\left(A_1B_2 - A_2B_1\right),\ f_1 = \psi_0\left(A_2D_1 - A_1D_2\right)/\left(A_1B_2 - A_2B_1\right).$$

It is necessary to satisfy last boundary condition on a wall

$$\tilde{\psi}(1) = \psi_1 + \psi_2 + \ldots + \psi_N = 0\,. \tag{25}$$

The condition (25) is intend for calculation of complex velocity $c = c_r + ic_i$ which is present in equations (16) – (18) only through complex $b = x + iy$. Here $x = \tilde{\alpha}c_i + \tilde{\alpha}^2/\mathrm{Re}^2$ and $y = \tilde{\alpha} + n\tilde{q} - \tilde{\alpha}c_r$ are real and imaginary parts b. The equation (25) is equivalent to two equations

$$G_R(x,y) = 0\,,\ G_I(x,y) = 0\,, \tag{26}$$

where G_R and G_I are real and imaginary parts of $\tilde{\psi}(1)$. The system of equations (26) was solved by iterations using the Newton – Rafson method. Let us assume that some iteration (x_n, y_n) and accordingly G_R^n, G_I^n are known. Expanding $G_R(x,y)$, $G_I(x,y)$ in Taylor series in a neighborhood of (x_n, y_n) we write

$$G_R^n + \frac{\partial G_R}{\partial x}\left(x_{n+1} - x_n\right) + \frac{\partial G_R}{\partial y}\left(y_{n+1} - y_n\right) = 0\,,\ G_I^n + \frac{\partial G_I}{\partial x}\left(x_{n+1} - x_n\right) + \frac{\partial G_I}{\partial y}\left(y_{n+1} - y_n\right) = 0 \tag{27}$$

From here we receive following iteration x_{n+1}, y_{n+1}. Partial derivatives in (27) are replaced by finite differences using small increments Δx, Δy, and the calculated increments ΔG_R, ΔG_I. Iterations has being stopped after reaching the conditions $\left|x_{n+1} - x_n\right| < 10^{-8}$, $\left|y_{n+1} - y_n\right| < 10^{-8}$.

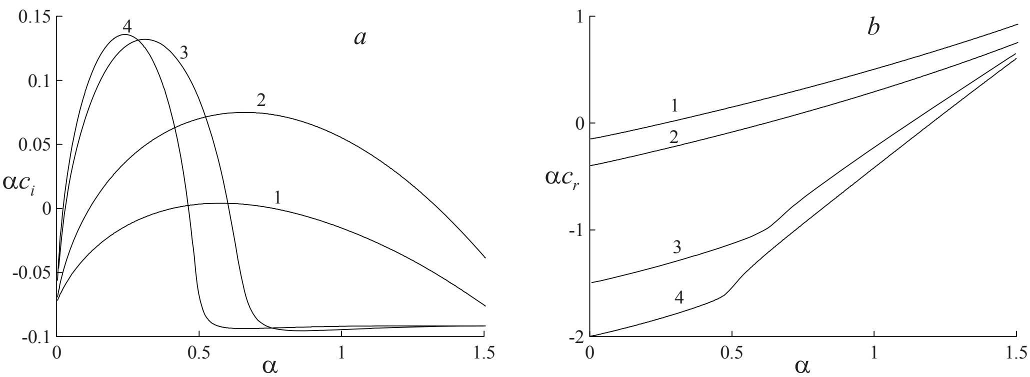

Fig. 1. An increment (*a*) and frequency of oscillations (*b*) versus wavenumber for Re = 200 and various values of a swirling parameter *q*: 1 – 0,15; 2 – 0,4; 3 – 1,5; 4 – 2

4. Results of calculations

Test calculations have shown that 35 – 40 terms of series are quite enough for approximation of required functions. Increase of an amount of terms up to 75 leads to a variation of the calculated values only in the sixth decimal sign. All calculations are made for a mode $n = -1$. Figure 1 presents increment and oscillation frequency versus wavenumber for Re = 200 and various values of a swirling parameter q.

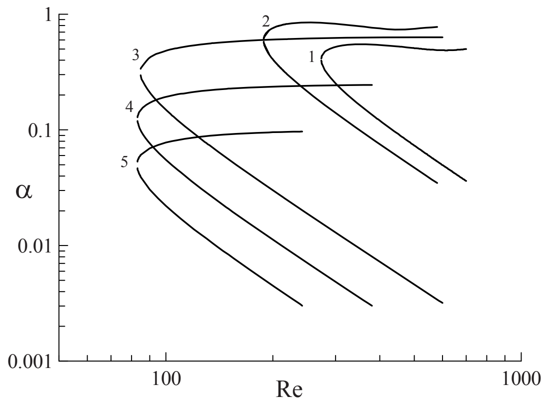

Fig. 2. Neutral curves for various values q: 1 – 0,15; 2 – 0,4; 3 – 1,5; 4 – 2

For $q < 0.1$ increment is negative for any wavenumber. Increasing of q gives range of wavenumber in which increment is positive (instability area). With growing q the instability area extends, and maximal increment grows. For $q > 1$ maxima of an increment increases slightly, and the instability area is narrowed down and is observed only in a range $0.03 < \alpha < 0.5$ for $q = 2$. The phase velocity of waves of maximum growth for small q positive, but is very small, and for $q > 1$ phase velocity of growing perturbations is negative.

Curves of a neutral stability for various swirling parameter are presented in Fig. 2 in coordinates (α, Re). For any fixed value q the instability area exists for $\mathrm{Re} > \mathrm{Re}_{cr}$. For small q the value Re_{cr} is great and monotonically decreases with growth of parameter of a curling ($\mathrm{Re}_{cr} \approx 27/q$). In a range $1 < q < 10$ the critical Reynolds number varies slightly ($\mathrm{Re}_{cr} \approx 83 \div 85$). In a limit $q \to \infty$ the calculated critical Reynolds number is 82.9201 that agree with value $\mathrm{Re}_{cr} = 82.92$ found earlier in [1, 5, 6]. The calculated boundary of stability area is shown in Fig. 3 (instability area lies above the curve). The calculated Re_{cr} and the matching α_{cr} are presented in the Table 1 in comparison with date calculated in [1].

Table 1. Calculated Re_{cr} and α_{cr} in comparison with data from [1]

q	0.01	0.02	0.06	0.1	0.2	0.6	1	10
Re_{cr}, our calculation	2696.47	1348.74	451.15	273.54	150.16	93.442	86.925	82.93
Re_{cr}, data [1]	2696.06	1347.82	450.97	273.57	150.14	93.44	86.92	82.92
α_{cr}, our calculation	0.0396	0.079	0.241	0.405	0.716	0.635	0.448	0.049
α_{cr}, data [1]	0.0400	0.079	0.241	0.404	0.716	0.634	0.447	0.050

As we see, our method gives good coincidence with the results given in [1] calculated by the numerical integration of the equations (2) – (5). It is necessary to note that the presented method of expansion of required functions into series is suitable not only for Poiseuille flow in a rotating pipe, but also for other vortex flows, for example for Batchelor vortex with axial and azimuthally velocity components $U = \exp(-\alpha r^2)$,

$W = q(1 - \exp(-\alpha r^2))/r$. Here the parameter α defines a vortex core size. Spreading out an exponent into the series, the velocity components can be presented as power series of variable r:

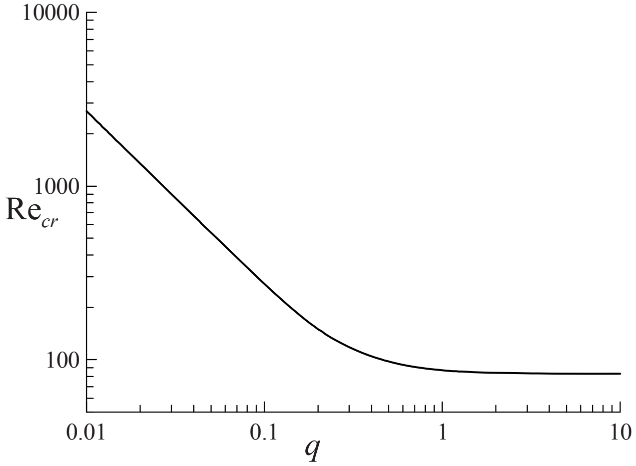

Fig. 3. Calculated boundary of stability area

$$U = 1 - \alpha r^2 + \frac{\alpha^2 r^4}{2} + \cdots,\ W = q\left(\alpha r - \frac{\alpha^2 r^3}{2} + \frac{\alpha^3 r^5}{6} + \cdots\right), \tag{28}$$

As seen, like in the case of Poiseuille flow, here are present only even powers of r for an axial component, and only odd powers r for an azimuthal one. Substituting (28) in (6) – (9), we get the expansions of required functions into series. Recurrent relations for series coefficients will be in this case more bulky, but the method of solution will be the same.

5. Conclusions

The new method is developed for stability analysis of swirl flows of viscous incompressible liquid in the cylindrical channel. The method is based on expansion of required functions into ascending power series on radial coordinate. The algorithm of the solution is reduced to iterative procedure in which series coefficients are calculated from system of the linear algebraic equations. Thereby in eigenvalue problem it is possible to avoid the difficulties connected with a numerical integration of system of the differential equations with singular point. As an example the stability of Poiseuille flow in a rotating pipe is considered. The new method agrees well with known numerical integration result.

Acknowledgements

This work was supported by the Russian Foundation for Basic Research, project № 10-08-01096-a and President Program for Support of the Leading Scientific Schools, project NSh-6686.2012.8.

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